Arithmetical Structures on Graphs

Darren Glass

Gettysburg College

(Joint work with various folks)

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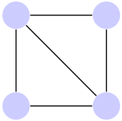

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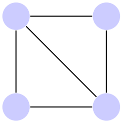

Let A be the adjacency matrix of the graph. Let D be an $n \times n$ diagonal matrix where diagonal entries are the nonnegative integers $\mathbf{d} = (d_1, d_2, d_3, d_4)$. Let $M = D - A$.

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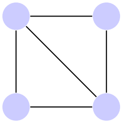

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If there exists a vector $\mathbf{r} = (r_1, \ldots, r_n)$ so that:

- All r_i are positive integers
- $gcd(r_1, \ldots, r_n) = 1$
- $\mathbf{r} \cdot M = \overrightarrow{0}$

then we say that $(\mathbf{r}, \mathbf{d})$ is an arithmetical structure on G.

Note that for any $\mathbf{d}$, M will have rank $n-1$ so $\mathbf{r}$ is unique if it exists.

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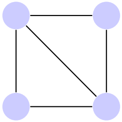
Example



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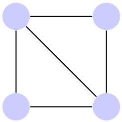

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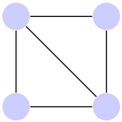

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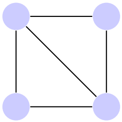

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$((1, 1, \ldots, 1), \mathbf{d})$ is an arithmetical structure on G.

Parenthetical Aside

For all graphs G, we have that $(L, (1, \ldots, 1))$ is an arithmetical structure on G.

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Definition

Let $\mathcal{D} = \{(d_1, \ldots, d_n) | \sum d_i = 0\}$. The Jacobian of G is defined to be $\mathcal{D}/Im(L)$.

Analogous to the Jacobian of a curve in algebraic geometry, and studied by many people (Bak, Baker-Norine, Propp, . . .) under many different names (Critical Groups, Sandpile Groups, . . .).

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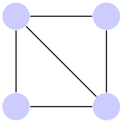
For any arithmetical structure $(\mathbf{r}, \mathbf{d})$ on G we can define an analogue to the Jacobian by considering $Ker(\mathbf{r})/Im(M)$. Can think of this as extending the analogy to curves with components of higher multiplicity.

Another Example



Let $M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 11 & -1 \\ -1 & 0 & -1 & 1 \end{bmatrix}$ and $\mathbf{r} = (3,4,1,4)$.

Then $\mathbf{r} \cdot M = \overrightarrow{0}$

$\Rightarrow (\mathbf{r}, (3,1,11,1))$ is an arithmetical structure on G.

What Does This Have To Do With Number Theory?

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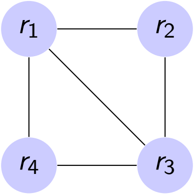
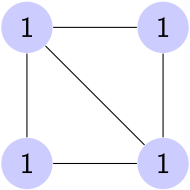
Definition

An arithmetical structure on G is a labeling of the vertices of G with nonnegative integers r_i so that $gcd(r_1, \ldots, r_n) = 1$ and for each vertex i we have that $r_i | \sum_{(i,j) \in E(G)} r_j$



An arithmetical structure is a vector (r_1, r_2, r_3, r_4) so that:

- $r_1 | r_2 + r_3 + r_4$
- $r_2 | r_1 + r_3$
- $r_3 | r_1 + r_2 + r_4$
- $r_4 | r_1 + r_3$



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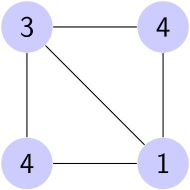

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Question

For a fixed graph, how many arithmetical structures are there?

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Theorem (Lorenzini)

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For small graphs, use Mathematica!

```
In[88]:= Do[
  Do[Do[Do[If[GCD[a, b, c, d] == 1, If[Divisible[b + c + d, a], If[Divisible[a + c, b], If[
        Divisible[a + c, d], If[Divisible[a + b + d, c], AppendTo[S, {a, b, c, d}]]]]]],
    {a, 1, 50}], {b, 1, 50}], {c, 1, 50}], {d, 1, 50}]

In[89]:= S

Out[89]= {{1, 1, 1, 1}, {3, 1, 1, 1}, {1, 2, 1, 1}, {5, 3, 1, 1}, {3, 4, 1, 1}, {2, 1, 2, 1},
  {4, 1, 2, 1}, {8, 5, 2, 1}, {1, 1, 3, 1}, {3, 2, 3, 1}, {1, 4, 3, 1}, {7, 10, 3, 1},
  {2, 1, 4, 1}, {6, 1, 4, 1}, {8, 3, 4, 1}, {14, 9, 4, 1}, {1, 3, 5, 1}, {4, 1, 6, 1},
  {3, 10, 7, 1}, {4, 3, 8, 1}, {2, 5, 8, 1}, {18, 5, 12, 1}, {4, 9, 14, 1},
  {12, 5, 18, 1}, {1, 1, 1, 2}, {1, 2, 1, 2}, {5, 2, 1, 2}, {7, 4, 1, 2}, {3, 1, 3, 2},
  {9, 4, 3, 2}, {1, 2, 5, 2}, {1, 4, 7, 2}, {3, 4, 9, 2}, {5, 1, 1, 3}, {5, 6, 1, 3},
  {4, 3, 2, 3}, {8, 1, 4, 3}, {2, 3, 4, 3}, {1, 1, 5, 3}, {1, 6, 5, 3}, {4, 1, 8, 3},
  {3, 1, 1, 4}, {7, 2, 1, 4}, {3, 4, 1, 4}, {11, 6, 1, 4}, {1, 1, 3, 4}, {9, 2, 3, 4},
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In[90]:= Length[S]

Out[90]= 63
```

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For a given family of graphs (paths, cycles, etc) are there better ways to count?

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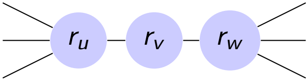
Lemma

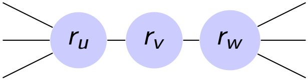
If v is a vertex of degree two which is connected to u and w, and $d|r_u$ and $d|r_v$ then $d|r_w$.



Proof: $r_v|(r_u + r_w) \Rightarrow d|(r_u + r_w) \Rightarrow d|r_w$

Lemma

If v is a vertex of degree two which is connected to u and w, and $r_v > r_u$ and $r_v > r_w$ then $r_v = r_u + r_w$.



Proof: $r_v | (r_u + r_w) < 2r_v \Rightarrow r_v = r_u + r_w$.

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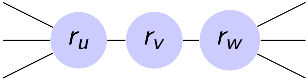
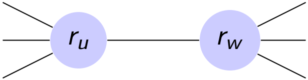


Proof: $r_v | (r_u + r_w) < 2r_v \Rightarrow r_v = r_u + r_w$.

Moreover, removing vertex v gives a valid arithmetical structure on a (smaller) graph:



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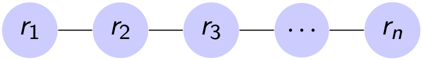

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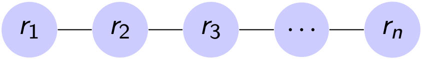

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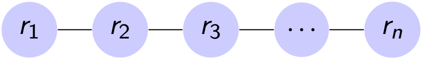

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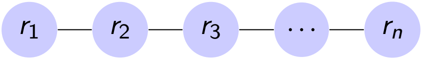

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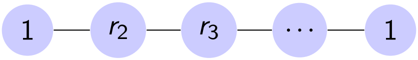
But $gcd(r_1, \ldots, r_n) = 1$, so we must have $r_1 = 1$. Similarly, $r_n = 1$.


1
r_2
r_3
$\cdots$
1

Note: No two consecutive entries are equal unless they are 1.



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We can repeat this process until we get all 1's.

Conversely...

Any arithmetical structure on P_n can be obtained by starting with the trivial structure on P_k for some $k \leq n$ and repeatedly adding vertices that are the sum of two adjacent vertices.

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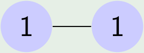
Example ($n = 2$)



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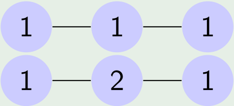

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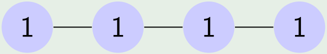

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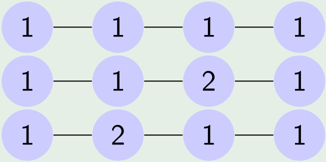

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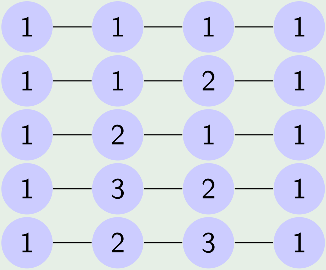

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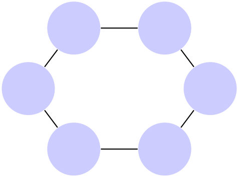
Theorem

The number of arithmetical structures on P_n is given by the Catalan number

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

Let C_n be a cycle on n vertices.



Arithmetical Structures on Cycles

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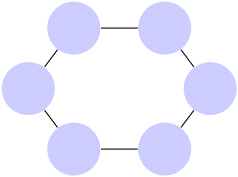
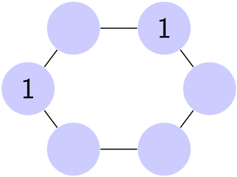


Similar story works. Any two consecutive elements must be relatively prime. So either all entries are equal to one or there is a local maximum that is the sum of its neighbors.

Arithmetical Structures on Cycles

Any structure can be attained by putting the number one in a positive number of vertices, and then filling others in with the sum of their neighbors.



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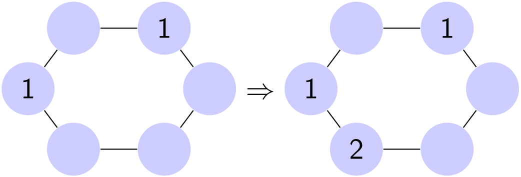

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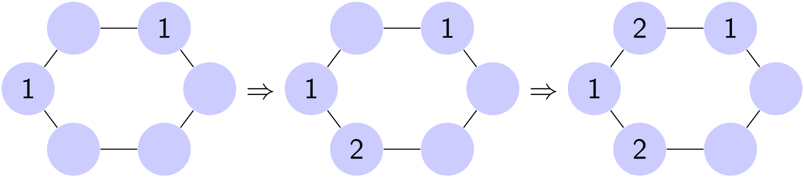

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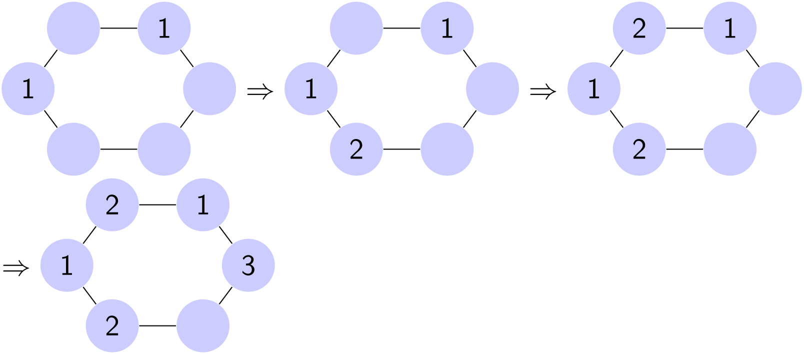

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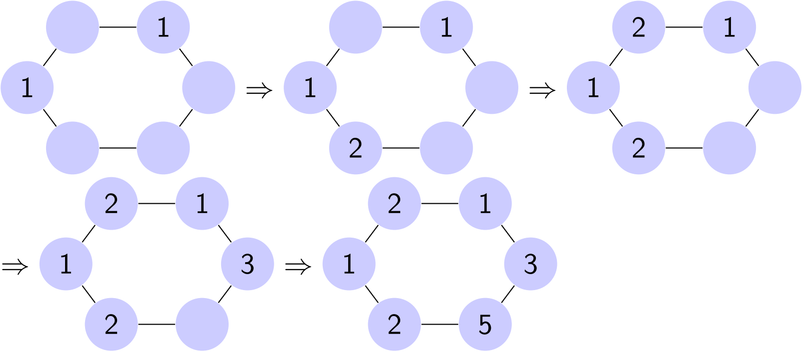

Theorem

The number of arithmetical structures on C_n with k ones is

$$\left(\!\!\binom{n}{n-k}\!\!\right) = \binom{2n-k-1}{n-k}$$

In particular, the total number of arithmetical structures on C_n is

$$\sum_{k=1}^{n} \left(\!\!\binom{n}{n-k}\!\!\right) = \left(\!\!\binom{n+1}{n-1}\!\!\right) = \binom{2n-1}{n-1}$$

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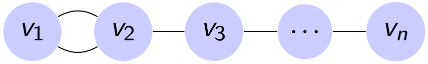
Let Q_n be the path on n vertices with first edge doubled:



The number of Arithmetical Structures on Q_n is $4C_{n-1} - 2C_{n-2}$.

Things get much more complicated when vertices have higher degrees.

Let Q_n be the path on n vertices with first edge doubled:

The number of Arithmetical Structures on Q_n is $4C_{n-1} - 2C_{n-2}$.

If first edge is tripled, there are:

$$9C_{n-1} - 9C_{n-2} - \frac{9}{n-1}\binom{2n-6}{n-2} - \frac{4}{n-1}\binom{2n-7}{n-2} - \frac{5}{n-1}\binom{2n-8}{n-2} - \frac{6}{n-1}\binom{2n-9}{n-2} - \frac{7}{n-1}\binom{2n-10}{n-2}$$

Approach With Josh Wagner (Gettysburg ’19)

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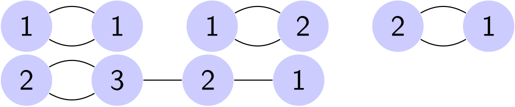
Seems as though it is easier to count these and then separately count the ways they can be subdivided to give all structures.

Example

- The only smooth structures on graphs Q_k are:



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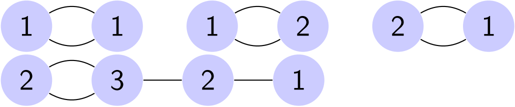

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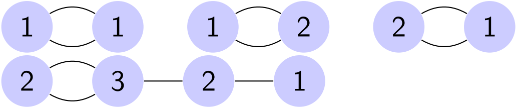

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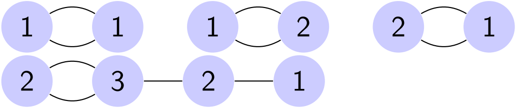

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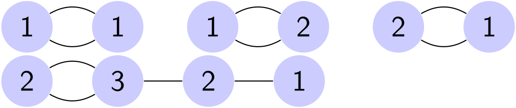

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This approach works for other families of graphs but appears to get very ugly very quickly.

Reasons I like this topic

- Intermingling of additive number theory and multiplicative number theory.
- Gives a way to introduce topics in modular arithmetic while working on open research problems.
- Connects number theory to linear algebra and algebraic geometry.
- Very accessible to both computer experimentation and exploration 'by hand'.
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