Gödel’s Theorem: Limits of logic and computation

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Overview

- Kurt Gödel, 1931, at age 25, in Vienna, shook up the foundations of mathematical thought as developed in Russell and Whitehead’s *Principia Mathematica*
- He showed that no formal axiomatic system of logic could be both *consistent* and *complete*
- Since predicate calculus was proven consistent, Gödel’s theorem showed that some truths are unprovable in the system

General-purpose definitions

- *Consistency:* Attribute of a system in which no false assertion can be proven $\forall p\ (\vdash p \Rightarrow p)$
- *Completeness:* Attribute of a system in which every true assertion can be proven $\forall p\ (p \Rightarrow\ \vdash p)$
- *Gödel number:* A value that uniquely encodes a logical assertion or sequence of assertions; e.g., a proof

Intuition of Gödel's proof

- Gödel needed a counter-example to the claim that a system can be both complete and consistent
- He showed that *something like* the following is true but unprovable: "*This assertion has no proof*"
- But the assertion actually used has a precise meaning

Definitions related to theorem

- *Proves(x, y, z):* The assertion that the sequence of symbols with Gödel number *x* is a valid proof that $S_y(z)$ is true, where S_y is the formula with Gödel number *y*, and *z* is a natural number.
- Let *G*(*y*) be the assertion that $\neg\exists x$ *Proves*(*x*, *y*, *y*) i.e., *G*(*y*) means that what the formula with Gödel number *y* asserts about *y* is unprovable
- *Example*: Where *y* encodes the predicate *even*, *G*(*y*) is the assertion that one cannot prove that the encoding of the predicate *even* is an even number

Gödel's theorem

"$\neg\exists x$ *Proves* (*x*, #(*G*), #(*G*))" is true in predicate calculus, but cannot be proven in the predicate calculus

- *Or:* What *G* asserts about predicate *G* is true but unprovable
- *Or:* There is no proof that *G*(#(*G*)) is true, but *G*(#(*G*)) is nevertheless true

Proof of Gödel's theorem

1. Suppose the theorem's claim is false and there exists a proof in predicate calculus that $\neg\exists x\ Proves(x, \#(G), \#(G))$.
2. Then the claim is true, by consistency of predicate calculus.
3. Therefore by its own assertion it is unprovable in predicate calculus.
4. This contradicts step 1, hence claim holds.

Note: The above proof goes outside the formal system of predicate calculus.

Another way to see the proof

- Think of an infinite truth table, with all formulas on one variable enumerated down the left side and arguments to the formulas as column headers
- There is a *true* in column c of row r iff $f_r(c)$ is true
- Let q be the Gödel number of the formula that asserts the unprovability of its argument
- Then there must be a *true* in column q of row q, because $f_q(q) = false$ would say that formula q is not unprovable, hence provable, hence true, a contradiction
- But if $f_q(q) = true$, then $f_q(q)$ is both true and (as it claims) unprovable

Godel’s theorem and computer science

- Using the same *diagonal proof* method, Turing proved a special case of Gödel’s theorem
- Given a *computable* function *f*, executing an algorithm to compute $y = f(x)$ proves that $f(x) = y$
- But for some functions *f*, for some *x*, $y = f(x)$, but there is no proof that $f(x) = y$
- This is the same as saying that no algorithm computes *f* -- example: the Halting Problem
- Gödel’s work in *recursive function theory* matches Turing’s results with automata

Larger significance

- Gödel’s theorem was part of a series of results establishing some *limits of knowledge*, that also included
  - Einstein’s relativity results
  - Heisenberg’s Uncertainty Principle
  - Turing’s theorem that some functions are uncomputable
- It marked the failure of Russell’s, Whitehead’s and Hilbert’s effort to reduce mathematics to first-order logic and set theory
- A category of *valid assertions* is established that we cannot *know* are valid

Sources

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