Godel's Theorem: Limits of logic and computation

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Overview

- Kurt Gödel, 1931, at age 25, in Vienna, shook up the foundations of mathematical thought as developed in Russell and Whitehead's *Principia Mathematica*
- He showed that no formal axiomatic system of logic could be both *consistent* and *complete*
- Since predicate calculus was proven consistent, Gödel's theorem showed that some truths are unprovable in the system

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General-purpose definitions

- *Consistency:* Attribute of a system in which no false assertion can be proven
  $\forall p(\,|\!-\, p \Rightarrow p)$
- *Completeness:* Attribute of a system in which every true assertion can be proven
  $\forall p(\, p \Rightarrow\, |\!-\, p)$

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Intuition of Gödel's counter-example

- Gödel showed that *something like* the following is true but unprovable:
  "*This assertion has no proof*"
- But the assertion actually used has a precise meaning

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Definitions related to theorem

- *Proof(x, y, z):* The sequence of symbols with Gödel number x is a valid proof that $S_y(z)$ is true, where S_y is the formula with Gödel number y, and z is a natural number.
- Let $G(y)$ be the assertion that $\neg\exists x$ Proof(x, y, y) i.e., $G(y)$ means that what the formula with Gödel number y asserts about y is unprovable
- *Example*: Where y is the predicate *even*, $G(y)$ is the (false) assertion that one cannot prove that the Gödel number of the predicate *even* is an even number

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Gödel's theorem

> $\neg\exists x$ Proof$(x, \#(G), \#(G))$ is true in predicate calculus, but cannot be proven in predicate calculus

- *Or:* What G asserts about predicate G is true but unprovable
- *Or:* There is no proof that $G(\#(G))$ is true, but $G(\#(G))$ is nevertheless true

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Proof of Gödel's theorem

1. Suppose the theorem's claim is false and there exists a proof in predicate calculus that $\neg\exists x$ Proof(x, #(G), #(G)).
2. Then the claim is true, by consistency of predicate calculus.
3. Therefore by its own assertion it is unprovable in predicate calculus.
4. This contradicts step 1, hence claim holds.

Note: The above proof goes outside the formal system of predicate calculus.

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What this has to do with computer science

- Computations and sets of computations are proofs about programs
- Gödel's theorem was precursor to Turing's result on the Halting Problem

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Larger significance

- Gödel's theorem was part of a series of results establishing limits of knowledge that also included
  - Einstein's relativity results
  - Heisenberg's Uncertainty Principle
- It marked the failure of Russell's, Whitehead's and Hilbert's effort to reduce mathematics to first-order logic and set theory

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Source

- Flake, *The computational beauty of nature*, MIT, 2001.

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