2. Propositional and predicate calculus[65]

Propositional calculus[66]

- It is a *language* of *assertions* that evaluate to *true* or *false*
- Assertion formulas include:
  - *true, false*
  - Symbols (*p, q, r,* ...) that may stand for assertions such as "roses are red"
  - Negations: $\neg p$
  - Conjunctions: $p \wedge q$
  - Disjunctions: $p \vee q$
  - Implications: $p \rightarrow \neg(q \vee \neg p)$

Semantics[67]

- Let *p, q,* be formulas
- Wherever p is true, $\neg p$ is false
- $(p \wedge q)$ is true when both *p* or *q* are true
- $(p \vee q)$ is true when either *p* or *q* or both are true
- $(p \rightarrow q)$ is true when *p* is false or *q* is true
- $(p \Leftrightarrow q)$ or $(p \equiv q)$ or (*p* iff *q*) is true when *p* and *q* have the same values under all interpretations (truth assignments)

Interpretations[68]

- An interpretation of a set of formulas is an assignment of truth values to the symbols
- *Example:* An interpretation of $(p \wedge \neg(p \vee q))$ is *p* = true, *q* = false. Under this interpretation, $(p \wedge \neg(p \vee q))$ is false
- A formula is *satisfiable* if it has an interpretation under which it is true

Truth tables[69]

- A truth table lists the values of a formula under all possible interpretations
- *Example:*

p	*q*	$(p \wedge \neg q)$
f	*f*	*f*
f	*t*	*f*
t	*f*	*t*
t	*t*	*f*

- Formulas with the same truth table are *equivalent*: for formulas ϕ and φ, $(\phi \Leftrightarrow \varphi)$ or $(\phi \equiv \varphi)$ or (ϕ iff φ)

Predicate calculus[70]

- Predicate calculus extends propositional calculus by enabling quantifiers and formulas in predicate (functional) form
- A predicate is a Boolean function
- *Examples:*
  $(\forall x)\ x < x + 1$ universal quantifier
  prime(5) 1-ary predicate
  $(\exists x)\ x > 1$ existential quantifier
- *Arity* of a predicate is its number of parameters

Variables in the predicate calculus[71]

- Suppose *x* is a variable in the domain of natural numbers
- In expressionis sum(*x*, 5), odd(*x*), prime(*x*), *x* is a *placeholder* that stands for an unknown *constant*
- "sum(*x*,5) = 8", "odd(*x*) = true" are meaningless without having a value for *x*
- "$x = 3 \rightarrow$ sum(x, 5) – 8", "$x = 1 \rightarrow$ odd(x) = true" are meaningful and true statements

Quantifiers[72]

- Meaningful:
  $(\exists x)\ x + 5 = 8$
  $(\forall x)\ x + 1 = x$
- Quantifiers *bind* variables so that they can be used meaningfully in sentences

Semantics of predicate calculus[73]

- In a real-world *domain*, there are relations:
  female(h_clinton) unary
  married(b_obama, m_obama) binary
- Predicates *female, married* here express these relations
- For a domain D, an *interpretation* over D is an assignment of entities in D to constants, assignment of non-empty subsets of D to variables, with each predicate mapping $D^n \rightarrow$ (t, f)

[65] Luger05
[66] Ibid.
[67] Ibid.
[68] Ibid.
[69] Ibid.
[70] Ibid.
[71] Ibid.
[72] Ibid.
[73] Ibid., p. 57

Truth value (semantics) of predicate-calculus expressions[74]

- Given expression E, and interpretation I for E over domain D, truth value is as in propositional calculus, in addition to:
- For sentence S,
  - $(\exists x)\ S$ is true iff some assignment under I has an assignment to x s.t. S is true
  - $(\forall x)\ S$ is true iff S is true for all assignments of values to x under I

Examples[75]

1. If it doesn't rain, Rose will go to the Park
¬weather(rain) → go(Rose, park)

2. Spark is a Spaniel and a good dog
isa(Spark, Spaniel) ∧ good-dog(Spark)

3. All basketball players are tall
$(\forall x)$ *(basketball-player(x) → tall(x))*

4. Nobody likes taxes
$\neg (\exists x)$ (likes(x, *taxes*))

Satisfiability, validity, consistency[76]

- An interpretation *satisfies* a sentence if it makes the sentence true.
- If I satisfies sentence S for every set of user assignments, then I is a *model* of S
- S or a set of sentences is *satisfiable* if some I satisfies it
- An unsatisfiable set of expressions is *unsatisfiable*
- If S is true for all interpretations, then S is *valid*
- $(\forall x)\ (P(x) \vee \neg P(x))$ is valid
- A sentence that is not true under any interpretation is *inconsistent*
- $(\forall x)\ (P(x) \wedge \neg P(x))$ is inconsistent

Inference[77]

- Expression *x logically follows* from a set S of sentences iff every interpretation that satisfies S also satisfies x
- *Inference rule:* A validity-maintaining procedure for deriving sentences from other sentences
- *Proof procedure*: An inference rule and an algorithm for applying it to a set of sentences to yield a new sentence

[74] Ibid., p. 58
[75] Ibid., p. 60
[76] Ibid., p. 62
[77] Idem.

Soundness and completeness[78]

- Sound: an inference rule that produces from set S only sentences that logically follow from S
- Complete: a rule that can infer from S every sentence that logically follows from S
- Examples:
  - Modus Ponens: $(p \wedge (p \rightarrow q)) \rightarrow q$
  - Modus Tollens: $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
  - And elimination, and introduction, universal instantiation

Unification[79]

- An algorithm for determining what substitutions are needed to make two sentences match
- *Cases for unify (E1, E2):*
  - E1 = E2: Return {} (no substitution)
  - E1 is variable: return {E2 / E1}
  - E2 is variable: return {E1 / E2}
  - Otherwise decompose E1, E2, return a composition of unification of the components

Predicate calculus and knowledge representation[80]

- Predicate calculus is a way to represent knowledge of
  - Objects
  - Relations
- Inference rules enable derivation of new knowledge

[78] Ibid., pp. 64-65
[79] Ibid., p. 66
[80] Ibid., p. 79