

## Theory uses definitions, examples, theorems, and proofs

Mathematics and theoretical computer science use precise *definitions* of concepts, which can often be clarified by intuitive *examples*. The results of work in theory are *theorems*, which are provable assertions about defined abstract objects and classes of objects. *Proofs* may be formal or informal. They rely on definitions as well as on logical rules of inference.

In theory of computing, we use constructive proofs often to reason about the behavior and computational power of models of computation, such as automata, and instances of models.

### Case: Structural induction

*Structural induction* is a proof method that uses induction on the size of a structure, such as a graph or tree, to show a property of all such structures. In the base case the property is easily shown for a small structure, and in the inductive case, it is shown that if all structures of a size  $n$  have the property, then all structures of size  $(n + 1)$  do as well. From this it follows by induction that all structures have the property.

**Definition:** a graph  $G$  is a tuple  $(V, E)$ , where  $V$  is a set of *vertices* and  $E \subseteq V \times V$  is a set of *edges*.

**Example:** a road map is a graph, where the towns and other places are vertices, and the roads joining towns are edges.

**Theorem:** For any graph  $G = (V, E)$ ,  $\sum_{v \in V} \delta(v) = 2|E|$

(The sum of the degrees of all vertices in a graph is twice the number of edges.)

#### Proof:

*Base step:*  $G = (\emptyset, \emptyset)$  (no edges, no vertices):  $\sum_{v \in V} \delta(v) = 2|E| = 0$

*Inductive step:* to show that if for all graphs  $G = (V, E)$  s.t.  $|V| = n$ ,  $\sum_{v \in V} \delta(v) = 2|E|$ , then the same holds for all graphs s.t.  $|V| = n + 1$ .

1. For any graph, adding one vertex alone (with no edges) changes neither  $\sum_{v \in V} \delta(v)$  nor  $E$ .
2. Adding  $k$  edges, yielding  $G' = (V, E')$ , increases the degree of *each* participating vertex by one:  $|E'| = |E| + 2k$ .
3. Hence  $G'$  has  $2k$  more edges than  $G$ .
4. So the sum of the degrees of the vertices in  $G'$  is still  $2|E|$ .