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Why induction and coinduction are important

Induction (recursion) is used to define functions that are computable by algorithms; it turns out that the recursively definable functions are precisely the ones that algorithms can compute. (Topic 5)

Induction also defines all kinds of finite structures like grammars and trees, and there are inductive proofs about a lot of these structures. (e.g., Topic 3) Any inductive definition has a base case. The definition of an inductively defined set may *use* itself; but such a set does not *contain* itself.

Peano's Axioms are an inductive definition of natural numbers:

(i) 0 is a natural number;
(ii) Every $n \in \mathbb{N}$ has a unique successor, n';
(iii) i and ii are the only ways to obtain a natural number

Arithmetic expressions using numerals, +, and ( ) are defined inductively:
expression → *numeral* | *numeral* + *expression* | (*expression*)

Coinduction defines infinite structures, including I/O streams and representations of real numbers. The fact that reals correspond to infinite bit streams will be central to the diagonal proof by Cantor (Topic 1) that there are "more" reals than natural numbers.

Whereas algorithms compute from finite input to finite output, interaction is inherently unbounded, so we deal with streams to characterize interaction (topic 6). These streams are defined coinductively.

A *non-well-founded set* may contain itself, directly or indirectly. *Example:* $A = \{ B, C \}$; $B = \{ A, D \}$. Non-well-founded sets violate the *well-foundedness axiom*, which specifies that the notion of a set containing or not containing itself is not meaningful. Non-well-founded sets use the *anti-foundation axiom*, which states that a circular definition of a set has a unique solution.

Streams are infinitely long sequences that may be defined coinductively in accordance with the anti-foundation axiom. Every stream of symbols is a symbol, followed by a stream of symbols. Streams contain streams, which contain streams, which contain streams…

Examples:

- The set of all bit streams $\{0,1\}^{¥}$ consists of any infinite sequence that consists of 0 or 1 followed by a stream.
- The set of streams over an alphabet, expressing ongoing processes such as interaction, may be defined coinductively as follows:
  $\Sigma^{\infty} = \{ ax \mid a \in \Sigma, x \in \Sigma^{\infty} \}$