Evaluating Peers' Arguments as the Catalyst for Learning in an Introduction to Proofs Course

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- Foundations of Higher Mathematics
- 3-credit "bridge" course between lower- and upper-division undergraduate mathematics courses
- The language of mathematics, set theory and proof, relations and functions, number systems, mathematical structures.
- 27 students
- Primarily sophomore and junior mathematics majors (including PSTs)

The Course

- Undergraduate students:
  - See the writing of proofs as a specific procedure that is to be replicated based on the text or instructor (e.g., Stylianou, Blanton, & Knuth, 2009).
  - View the mathematics instructor as the final / only arbiter for validating their mathematical argument (e.g., Harel & Sowder, 2007).

- Mathematicians:
  - *Negotiate* the validity of presented arguments within their community of practice (e.g., Inglis & Alcock, 2012; Inglis, Mejia-Ramos, Weber, & Alcock, 2013; Weber, 2008).

- Driving Question:
  - How can I move my students away from the view of the instructor as the final/only arbiter for validation of mathematical proof and toward the view that they possess the tools and sense-making capabilities to successfully construct and validate mathematical arguments?

The Problem

- Learner-generated examples (Watson & Mason, 2005)

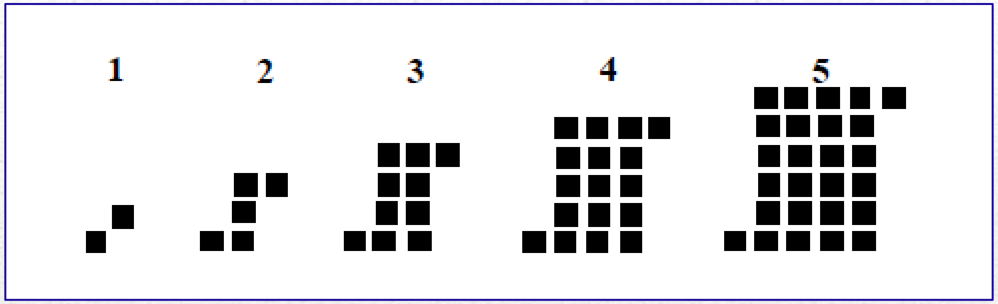
- Small-group learning (Kyndt, Raes, Lismont, Timmers, Cascallar, & Dochy, 2013; Springer, Stanne, & Donovan, 1999)

Research-Based Course Design

Typical Instructional Schedule

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
		Students work individually to make sense of “new” mathematics in assigned problem set.				Students electronically submit problem set.
Instructor reviews student work, identifies key misconceptions, and develops activities based on learner-generated examples.		In-class activity.		In-class activity. Assessment sheets returned.		
		Students submit revision.				

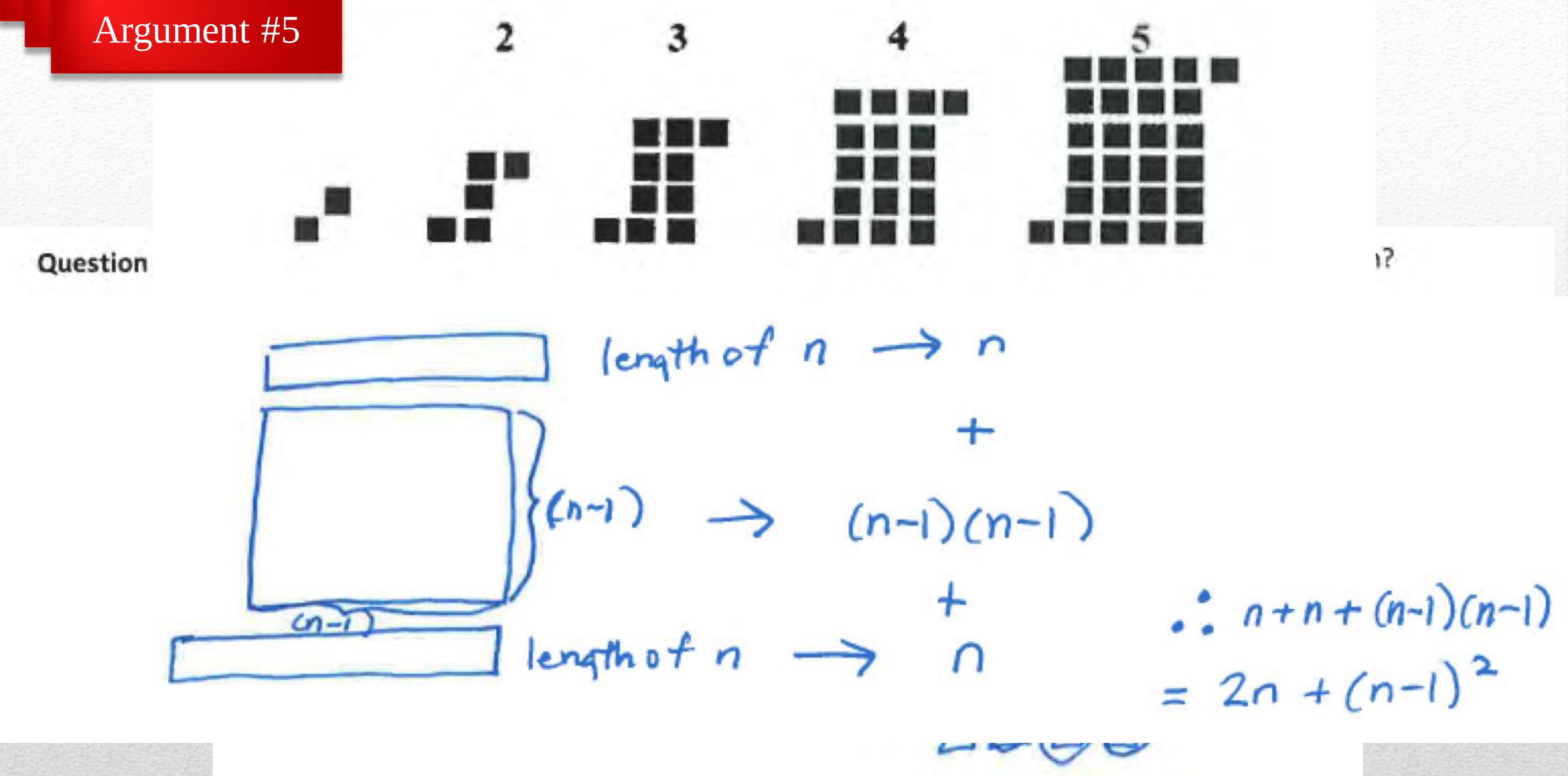
- **Question 1.** Consider the pattern below. How many square tiles would there be in the eighth step of this pattern?



- **Question 2.** Write an expression for the total number of square tiles in the figure at an arbitrary step (n) of the pattern.
- **Question 3.** Prove that your expression is a valid representation of the number of tiles at step n of the pattern. You may use drawings, words, numbers, and/or symbols for your proof.

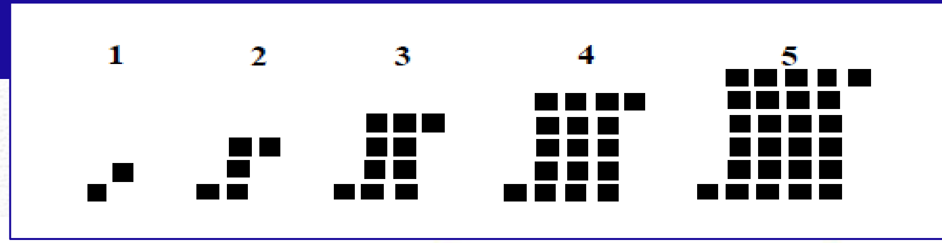
Week 1: Developing a Course Rubric for Proof-Writing

Argument #5



Instructional Goal: Multiple Ways to "View" the Generalization of a Pattern

Argument #2



In each step of the equation, we are given (n+1) rows and (n+1) columns. From the first and last columns, we are missing n number of squares. From this, we get the equation:

$$T = (n+1)(n+1) - 2n$$

This means that for each step, we get a square made up of (n+1) rows and columns missing n unit squares from each side of the bigger square, which is representative of the pattern shown above. By simplifying the equation we can see

$$T = (n+1)(n+1) - 2n = n^2 + 2n + 1 - 2n = n^2 + 2n - 2n + 1 = n^2 + 1.$$

Therefore, as long as the pattern holds, the equation can be represented by:

$$T = n^2 + 1.$$

Proof Criterion	Clear identification of parameters, constraints, and assumptions	Generalization	Chain of Evidence, Structure, and Clarity	Validity/Correctness
Descriptors	a1. Define the statement of the problem and the givens. a2. Define all variables. a3. Explain the boundaries of the solution (e.g., which numbers or number systems for which the proof works). a4. Make explicit all assumptions.	b1. Proof should apply to all situations/values within the specified parameters. b2. Answer the question "why?" b3. The conclusion should be stated in general terms. b4. Make connection between concrete examples and generalization	c1. Demonstrate reasoning that follows a logical sequence. c2. Argument is clear, complete, concise, and simplified. c3. Consider including elements to clarify argument, such as generalized visual representations, or concrete examples. c4. Make explicit use of definitions to aid in the precision and clarity of your argument.	d1. Proof conclusion is valid/correct/true

Student-Developed Course Rubric for Proof Writing

"It has given me a new look at proof. When I had taken the class the first time I was only exposed to the formal way and no other way of writing a proof. This has helped me see that there is not just one way to write a proof."

"I have also realized that there is not only one equation for every problem. What makes a good proof is being able to explain how one got the equation to work and why it works."

Student Reflections

"This has impacted my understanding of a proof by forcing me to think about what it takes to truly prove something. Before doing this engagement, I thought I would be able to prove something was true by showing an example of the solution working. Now I know it takes much more than examples to make a valid proof."

Student Reflections

“Sometimes words and lengthy descriptions can be improved or even replaced with well-designed visual aides. The focus on simplicity has also helped me understand the fine line between a well-written and structured chain of evidence and a poorly-written one.”

Student Reflections

Using learner-generated examples and small-group learning….

- ✧ **Helped students think about alternative approaches to "start" a proof.**
- ✧ **Motivated students to work from the definitions and given assumptions.**
- ✧ **Motivated students to be able to defend solutions using a clear and complete chain of evidence.**
- ✧ **Encouraged students to take ownership of their learning.**

Conclusion

- I would like to thank my colleagues who have been instrumental in helping me develop the ideas guiding my course design:
  - Justin D. Boyle, University of New Mexico
  - Yi-Yin (Winnie) Ko, Indiana State University
  - Sean P. Yee, California State University Fullerton

- My contact information:
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Thank You!

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