**LAB for *THE GEOMETER’S SKETCHPAD***

**THE NINE-POINT CIRCLE and THE EULER LINE**

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**Objectives:** 1) To create the **Nine-Point Circle** for an arbitrary ABC, and verify that it does contain the relevant 9 points (that is, the midpoints A***'***, B***'***, C***'*** of the three sides, the bases D, E, F of the three altitudes, and the points J, K, L midway between the orthocenter and the vertices).

2) To create the **Euler Line** for an arbitrary triangle and verify that that the circumcenter O, orthocenter H, centroid G, and center N of the Nine-Point Circle for the triangle are indeed collinear.

**A Quick Tour of *The Geometer’s Sketchpad***:

* + In *The Geometer’s Sketchpad*, the basic **tools** (operations) are available on the panel on the left side of the screen.
  + Use the “**arrow**” tool to **select** (that is, highlight) items. To **de-select** (that is, un-highlight) an object, simply click on it.
  + Use the “**dot**” (point) tool to create **points**. To **label** a point, select the point with the “arrow” tool, and then right-click and select “Label Point.” You can change a label by double-clicking on that label and then replacing the old label with a new one.
  + Use the “**line**” tool to create line segments between points by dragging the cursor from one of the points to the other.
  + Use “**Edit: Undo**” to undo the previous operation.
  + One of the most common errors when using *The Geometer’s Sketchpad* is trying to perform an operation on one or more highlighted objects when some other objects are accidentally highlighted as well. Whenever you run into such a problem, first de-select all objects, and then highlight only those objects that you want to have highlighted.

**Construction of The Nine-Point Circle:**

1. Create 3 noncollinear points on the screen (using the “dot” tool), and label them as A, B, and C. Then construct ABC by connecting each pair of points with a line segment (using the “line” tool).
2. Construct the three midpoints of the sides of ΔABC. (Highlight each side in turn and use: “Construct: Midpoint”.)

Label the midpoint of side BC as A***'***.

Label the midpoint of side AC as B***'***.

Label the midpoint of side AB as C***'***.

1. Construct a perpendicular from A to side BC (highlight both point A and side BC simultaneously, then use: “Construct: Perpendicular Line”).

Construct the intersection of side BC and the perpendicular line. (Highlight both segment BC and the perpendicular line together, then use: “Construct: Intersection.”)

Label this intersection as D.

Similarly, construct a perpendicular from B to side AC, and label the intersection as E.

1. Label the intersection of these two perpendiculars as H (**orthocenter**).

[The orthocenter H is the common intersection of the three altitudes of the triangle.]

Construct a perpendicular from C to side AB, and label the intersection as F.

[Of course, this perpendicular goes through H as well.]

1. Construct segment AH, and then construct its midpoint, and label it as J.

[This will convert the rest of the perpendicular through A into a dotted line.]

Similarly, construct the midpoint of segment BH, and label it as K.

Similarly, construct the midpoint of segment CH, and label it as L.

6. Construct segment JA***'***. Construct the midpoint of JA***'*** and label it as N (**center of the Nine-Point Circle**).

[At this point you can simplify the drawing by “hiding” the segment JA***'*** as well as the three perpendiculars created in steps 3 and 4 above. To do this, simply right-click on each one and select “Hide”. All of the points should still remain on the screen, however, including J, A***'***, and N.]

7. Highlight points N and A***'*** (in that order!), then Construct: Circle by Center+Point. You have created the **Nine-Point Circle** for ABC!

***Notice that this circle continues to go through all of the points A', B', C', D, E, F, J, K, L, even after points A, B, and/or C are moved about on the screen.*** (You can move a point about the screen by highlighting and dragging it.) (***Life doesn’t get much better than this!***)

***Have your instructor verify your work at this point!***

**Construction of The Euler Line:**

8. Draw the perpendicular to side CB through A***'***. (Highlight A***'*** and side CB

simultaneously, and then use: “Construct: Perpendicular Line”.)

Draw the perpendicular to side AC through B***'***.

Label the intersection of these two perpendiculars as O (**circumcenter**).

[The circumcenter O of a triangle is the common intersection of the three perpendicular bisectors of the sides of the triangle. We only need two of the perpendicular bisectors to locate O.]

9. Construct line segments AA***'*** and BB***'***.

Label the intersection of these line segments as G (**centroid**).

[The centroid G is the common intersection of the medians of the triangle. We only need two of the medians to locate G.]

[You can “hide” the perpendicular bisectors and medians created in these last two steps in order to simplify the drawing, but all of the points should remain, including O and G.]

10. Change the “line segment” tool to an “infinite-type line” tool (by dragging it to the right).

Draw an infinite line through H and O (**Euler Line**).

You have created the ***Euler Line*** for ABC!

[Notice that this line goes through all of the points H, N, G, and O.]

Then change the “line” tool back to a “(finite) segment-type line” tool.

11. [Finally, we will verify that the relative distances between points along the Euler Line are correct.]

Measure the distance from H to G, and the distance from G to O. (In each case, highlight the two endpoints simultaneously, and then use: “Measure: Distance.”) (These distances should be displayed in little textboxes on the screen.)

Calculate twice the length of segment GO by using “Number: Calculate”, then when the calculator appears on the screen, select “2”, then “\*”, then click on the newly created window for the length GO, and select “OK”. (This creates a window in which the current value of “2\*GO” is given.)

***Notice that HG = 2(GO), even when points A, B, and C are moved about on the screen.***

12. Measure the distance from H to O and the distance from H to N.

Calculate 2\* HN.

***Notice that HO = 2(HN), even when points A, B, and C are moved about on the screen.***

***Have your instructor verify your work at this point!***

# *Print out your final picture. Be sure that all of the points you constructed and their labels are clearly distinguishable from each other in the figure. (Move points A, B, or C, if necessary.) Turn in your final picture!*

***Extra Credit Fun:* Feuerbach’s Theorem**

Feuerbach’s Theorem states that the incircle as well as the three excircles of any triangle are all tangent to the Nine-Point Circle of the triangle.

13. Construct the internal angle bisector at A. (Highlight B, A, C in that order, then use: “Construct: Angle Bisector”.)

Similarly, construct the internal angle bisector at B.

Label the intersection of these internal angle bisectors as I (**incenter**).

[The incenter I is the common intersection of the three internal angle bisectors. It is the center of a circle that is tangent (interior-wise) to all three sides of the triangle.]

14. Draw the perpendicular from I to BC, and label the intersection as Z.

Highlight points I and Z (in that order) to draw a circle with center I and radius IZ. (**incircle**)

***Notice that the incircle remains tangent to the Nine-Point Circle (even as A, B, C are moved), as predicted by Feuerbach’s Theorem.***

15. Draw an “infinite” line through A and B.

Similarly, draw an “infinite” line through A and C.

Use the “point” tool to create (and then label) a point T on extended AB so that angle CBT is an external angle at B.

Draw an “infinite” line through A and I.

Bisect external angle CBT, and label the intersection of this bisector and line AI as X (an **excenter**).

[There are three excenters for a triangle. Each excenter is the common intersection of two external angle bisectors and the internal angle bisector of the remaining angle. Each excenter is the center of a circle that is tangent (exterior-wise) to the triangle. Notice how the constructed excircle is tangent to side BC as well as both of the (extended) sides AB and AC.]

16. Draw the perpendicular from X to BC.

Label the intersection of BC and this perpendicular as U.

Highlight points X and U (in that order) to draw the circle with center X and radius XU. (This is an **excircle.**)

Notice that this excircle is tangent to the Nine-Point Circle (even as A, B, C are moved), as predicted by Feuerbach’s Theorem.

***Have your instructor verify your work at this point!***