Basic Counting Techniques

- What is the Multiplication Principle?
- What is a factorial?
- What is a permutation?
- What is a combination?

Example

- Suppose you have eight shirts, four sweaters, and five pair of pants, all of which match in color so that any shirt, sweater, and pants can be worn together. How many outfits consisting of a shirt, a sweater, and a pair of pants can you make?

Example

- Suppose you have eight shirts, four sweaters, and five pair of pants, all of which match in color so that any shirt, sweater, and pants can be worn together. How many outfits consisting of a shirt, a sweater, and a pair of pants can you make?

Shirt: You have eight choices.

Example

- Suppose you have eight shirts, four sweaters, and five pair of pants, all of which match in color so that any shirt, sweater, and pants can be worn together. How many outfits consisting of a shirt, a sweater, and a pair of pants can you make?

Sweater: You have four choices.

Example

- Suppose you have eight shirts, four sweaters, and five pair of pants, all of which match in color so that any shirt, sweater, and pants can be worn together. How many outfits consisting of a shirt, a sweater, and a pair of pants can you make?

Pants: You have five choices.

Example

- Suppose you have eight shirts, four sweaters, and five pair of pants, all of which match in color so that any shirt, sweater, and pants can be worn together. How many outfits consisting of a shirt, a sweater, and a pair of pants can you make?

Outfits: You have $8 \cdot 4 \cdot 5$ possible outfits.

Example

- Suppose you have eight shirts, four sweaters, and five pair of pants, all of which match in color so that any shirt, sweater, and pants can be worn together. How many outfits consisting of a shirt, a sweater, and a pair of pants can you make?

Outfits: You have 160 possible outfits.

Multiplication Principle

- Suppose you have a sequence of n choices to make and there are
  - M_1 ways to make choice 1,
  - M_2 ways to make choice 2,

and so on with

  - M_n ways to make choice n.
- There are $M_1 \cdot M_2 \cdots M_n$ ways in which to make the sequence of n choices.

Example

- How many license plates can be made using three digits followed by three letters?

Example

- How many license plates can be made using three digits followed by three letters?
- Note: The question does not state that we cannot reuse/repeat the digits or the letters.

Example

- How many license plates can be made using three digits followed by three letters?
- With replacement, there are always ten digits available and twenty-six letters available.

Example

- How many license plates can be made using three digits followed by three letters?
- What are the digits?

Example

- How many license plates can be made using three digits followed by three letters?
- The digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Example

- How many license plates can be made using three digits followed by three letters?
- The digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
- There are ten possible digits.

Example

- How many license plates can be made using three digits followed by three letters?
- There are $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26$ different license plates possible.
- There are $10^3 \cdot 26^3$ different license plates possible.
- There are 17,576,000 different license plates possible.

Example

- How many license plates can be made using three digits followed by three letters if the letters cannot be repeated?

Example

- How many license plates can be made using three digits followed by three letters if the letters cannot be repeated?
- If the letters cannot be repeated then there are 26 letters available for the first letter, 25 letters available for the second letter, and 24 letters available for the third letter.

Example

- How many license plates can be made using three digits followed by three letters if the letters cannot be repeated?
- There are $10 \cdot 10 \cdot 10 \cdot 26 \cdot 25 \cdot 24$ different license plates possible.
- There are $10^3 \cdot 26 \cdot 25 \cdot 24$ different license plates possible.
- There are 15,600,000 different license plates possible.

Example

- How many license plates can be made using three digits followed by three letters if the digits cannot be repeated?

Example

- How many license plates can be made using three digits followed by three letters if the digits cannot be repeated?
- If the digits cannot be repeated then there are 10 digits available for the first digit, 9 digits available for the second digit, and 8 digits available for the third digit.

Example

- How many license plates can be made using three digits followed by three letters if the digits cannot be repeated?
- There are $10 \cdot 9 \cdot 8 \cdot 26 \cdot 26 \cdot 26$ different license plates possible.
- There are $10 \cdot 9 \cdot 8 \cdot 26^3$ different license plates possible.
- There are 12,654,720 different license plates possible.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated?

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated?
- If the digits cannot be repeated then there are 10 digits available for the first digit, 9 digits available for the second digit, and 8 digits available for the third digit.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated?
- If the letters cannot be repeated then there are 26 letters available for the first letter, 25 letters available for the second letter, and 24 letters available for the third letter.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated?
- There are $10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24$ different license plates possible.
- There are 11,232,000 different license plates possible.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?
- There are nine non-zero digits available for the first digit.
- There are 25 non-O letters available for the first letter.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?
- There are nine digits remaining for the second digit; eight of them are non-zero and the ninth is zero.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?
- There are twenty-five letters remaining for the second letter; twenty-four of them are non-O and the twenty-fifth is the letter O.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?
- There are eight digits remaining for the third digit.

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?
- There are twenty-four letters remaining for the third letter

Example

- How many license plates can be made using three digits followed by three letters if neither letters nor digits can be repeated, if the first digit cannot be zero, and if the first letter cannot be O?
- There are $9 \cdot 9 \cdot 8 \cdot 25 \cdot 25 \cdot 24$ different license plates possible.
- There are 9,720,000 different license plates possible.

Example

- In how many ways can a President, Vice-President, Secretary, and Treasurer be elected for a club having fifty members?

Example

- In how many ways can a President, Vice-President, Secretary, and Treasurer be elected for a club having fifty members?
- Since each member can only hold one office in the club, the election is made without replacement.

Example

- In how many ways can a President, Vice-President, Secretary, and Treasurer be elected for a club having fifty members?
- We assume that each member of the club is willing and able to meet the responsibilities of serving as a club officer.

Example

- In how many ways can a President, Vice-President, Secretary, and Treasurer be elected for a club having fifty members?
- There are $50 \cdot 49 \cdot 48 \cdot 47$ ways in which to elect the officers.
- There are 5,527,200 ways in which to elect the officers.

Example

- In how many ways can one arrange seven books on a shelf?

Example

- In how many ways can one arrange seven books on a shelf?
- Since a book can only be placed on the shelf once, the books are placed without replacement.
- All the books are being placed on the shelf.

Example

- In how many ways can one arrange seven books on a shelf?
- There are seven choices for the first book, six choices for the second book, five choices for the third book, four choices for the fourth book, three choices for the fifth book, two choices for the sixth book, and one choice for the seventh book.

Example

- In how many ways can one arrange seven books on a shelf?
- There are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways in which to order the books on the shelf.
- There are 5040 ways in which to order to order the books on the shelf.

Example

- In how many ways can one arrange seven books on a shelf?
- There are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways in which to order the books on the shelf.
- Since we order all the books, there are $7!$ ways in which to order the books on the shelf.

What is 7!?

- 7! is seven factorial.

What is n!?

- n!, read n factorial, is the product of all the counting numbers less than or equal to n.

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots (3) \cdot (2) \cdot (1)$$

What is n!?

- n!, read n factorial, is the product of all the counting numbers less than or equal to n.

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots (3) \cdot (2) \cdot (1)$$

- n! is the number of ways in which to order n objects.

Combination

- We use a combination to determine the number of groups without order that one can make from a collection of objects.

Combination

- Notation: $_nC_r$ or $C(n,r)$.
- Note: We will use $C(n,r)$.
- $C(n,r) = n!/[(n - r)! \cdot r!]$

Combination

- **Notation:** ${}_{n}C_{r}$ or $C(n,r)$.
- **Note:** We will use $C(n,r)$.

$$C(n,r) = \frac{n!}{(n-r)!\ r!}$$

Combination

- $C(n,r)$ is the number of ways in which n objects can be taken r at a time without order.
- $C(n,r)$ is the number of groups of size r that can be made without order from a collection of n objects.
- $C(n,r)$ is the number of ways to take n things r at a time without order.

Permutation

- We use a permutation to determine the number of groups with order that one can make from a collection of objects.

Permutation

- Notation: $_{n}P_{r}$ or $P(n,r)$.
- Note: We will use $P(n,r)$.
- $P(n,r) = n!/(n - r)!$

Permutation

- Notation: $_nP_r$ or P(n,r).
- Note: We will use P(n,r).

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutation

- P(n,r) is the number of ways in which n objects can be taken r at a time with order.
- P(n,r) is the number of groups of size r that can be made with order from a collection of n objects.
- P(n,r) is the number of ways to take n things r at a time with order.

Example

- In how many ways can you select five of your twelve favorite CD's to take with you to the office?

Example

- In how many ways can you select five of your twelve favorite CD's to take with you to the office?
- There is no order since we are not interested in actually playing the CD's.

Example

- In how many ways can you select five of your twelve favorite CD's to take with you to the office?
- n = 12 since this is the size of the collection, your favorite CD's
- r = 5 since this is the number of CD's that you want to take to the office.

Example

- In how many ways can you select five of your twelve favorite CD's to take with you to the office?
- There are $C(12,5)$ ways in which to select five of your twelve favorite CD's to take to the office.

How do we calculate C(12,5)?

- $C(12,5) = 12!/[7!\ 5!]$
- **Note:**
  $12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
  $= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!$
- $C(12,5) = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!/[7!\ 5!]$
  $= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8/5!$
  $= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
  $= 11 \cdot 9 \cdot 8$
  $= 792$

Example

- In how many ways can you select five of your twelve favorite CD's to take with you to the office?
- There are 792 ways in which to select five of your twelve favorite CD's to take to the office.

Example

- In how many ways can you play three of your twelve favorite CD's to take with you to the office?

Example

- In how many ways can you play three of your twelve favorite CD's to take with you to the office?
- Since we are interested in playing the CD's, different orderings of the CD's creates different play lists. So, order matters for this example and we use a permutation.

Example

- In how many ways can you play three of your twelve favorite CD's to take with you to the office?
- There are P(12,3) ways in which to play three of your twelve favorite CD's to take to the office.

How do we calculate P(12,3)?

- P(12,3) = 12!/9!
- **Note:**
  $12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
  $= 12 \cdot 11 \cdot 10 \cdot 9!$
- $P(12,5) = 12 \cdot 11 \cdot 10 \cdot 9!/9!$
  $= 12 \cdot 11 \cdot 10$
  $= 1,320$

Example

- In how many ways can you play three of your twelve favorite CD's to take with you to the office?
- There are 1,320 ways in which to play three of your twelve favorite CD's to take to the office.

Example

- How many five card hands containing only face cards can be dealt from a fair poker deck?

Example

- How many five card hands containing only face cards can be dealt from a fair poker deck?
- We use a combination to determine the number of card hands since card hands have no order. That is, no matter how one orders the cards that (s)he holds, the hand is the same.

Example

- How many five card hands containing only face cards can be dealt from a fair poker deck?
- There are 12 face cards available.
- There are five cards in the hand.
- The number of possible hands is $C(12,5)$.

Example

- How many five card hands containing only face cards can be dealt from a fair poker deck?
- There are 12 face cards available.
- There are five cards in the hand.
- There are 792 five-card hands consisting of only face cards.

Important Note

- Card hands have no order.
- Cards draw one at a time have order.

Example

- How many five card hands can be dealt from a fair poker deck?

Example

- How many five card hands can be dealt from a fair poker deck?
- We use a combination since card hands have no order.

Example

- How many five card hands can be dealt from a fair poker deck?
- There are $C(52,5)$ five-card hands.
- 2,598,960 five-card hands can be dealt from a fair poker deck.

How do you get 2,598,960?

$C(52,5) = 52!/[47!\ 5!]$

$= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!/[47!\ 5!]$

$= [52 \cdot 51 \cdot 50 \cdot 49 \cdot 48]/5!$

$= [52 \cdot 51 \cdot 50 \cdot 49 \cdot 48]/[5 \cdot 4 \cdot 3 \cdot 2 \cdot 1]$

$= 52 \cdot 17 \cdot 10 \cdot 49 \cdot 6$

$= 2598960$

Probability

- Combinations, permutations, the multiplication principle, and factorials can be used in calculating probability.
- Keep in mind that each of these produces a number.
- The number of card hands of a particular type, for example, can be the number of simple events of that type.

Example

- **What is the probability that a five card hand dealt from a fair poker deck contains two numbered cards and three lettered cards?**

Example

- What is the probability that a five card hand dealt from a fair poker deck contains two numbered cards and three lettered cards?
- We use three combinations, one for the number of groups of two numbered cards, one for the number of three lettered cards, and one for the number of five-card hands.

Example

- What is the probability that a five card hand dealt from a fair poker deck contains two numbered cards and three lettered cards?
- C(36,2) is the number of groups without order of two numbered cards.

Example

- What is the probability that a five card hand dealt from a fair poker deck contains two numbered cards and three lettered cards?
- C(16,3) is the number of groups without order of three lettered cards.

Example

- What is the probability that a five card hand dealt from a fair poker deck contains two numbered cards and three lettered cards?
- C(52,5) is the number of groups without order of five cards.

Example

- What is the probability that a five card hand dealt from a fair poker deck contains two numbered cards and three lettered cards?
- P(two numbered, three lettered) =

$$\frac{C(36,2)\ C(16,3)}{C(52,5)}$$