Conditional Probability

- What is conditional probability?
- What is the Multiplication Rule?

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- For example, what is the probability that event E occurs *if we know that* event F occurs?

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- The simplest way in which to define conditional probability is as probability with conditions.
- For example, what is the probability that event E *given* event F occurs?

Notation for Conditional Probability

- $P(E|F)$ denotes the probability that event E occurs given event F occurs

Calculating Conditional Probability

- The key to determining $P(E|F)$ is knowing that we are interested in the probability that event E occurs, knowing that event F occurs.
  - Since we know that event F occurs, we treat event F as the sample space and use it to determine the denominator for the calculation.

Calculating Conditional Probability

- The key to determining $P(E|F)$ is knowing that we are interested in the probability that event E occurs, knowing that event F occurs.
  - Since we want event E to occur while event F occurs, we use the event $E \cap F$ to determine the numerator for the calculation.

Calculating Conditional Probability

- $P(E|F)$ is the quotient of the number of simple events in the event $E \cap F$ and the number of simple events in the event F.

Calculating Conditional Probability

- Letting $n(E \cap F)$ be the number of simple events in the event $E \cap F$ and letting $n(F)$ be the number of simple events in the event F,

$$P(E|F) = n(E \cap F) / n(F)$$

Calculating Conditional Probability

- Given $P(E|F) = n(E \cap F) / n(F)$ and dividing the numerator and the denominator by $n(S)$, the number of simple events in the sample space, S, for the experiment,
  - The numerator becomes

$$n(E \cap F) / n(S)$$

but $n(E \cap F) / n(S) = P(E \cap F)$

Calculating Conditional Probability

- Given $P(E|F) = n(E \cap F) / n(F)$ and dividing the numerator and the denominator by $n(S)$, the number of simple events in the sample space, S, for the experiment,
  - The denominator becomes

    $$n(F) / n(S)$$

    but $n(F) / n(S) = P(F)$

Calculating Conditional Probability

- Given $P(E|F) = n(E \cap F) / n(F)$ and dividing the numerator and the denominator by $n(S)$, the number of simple events in the sample space for the experiment,
- $P(E|F)$ can be re-expressed as

$$P(E|F) = P(E \cap F) / P(F)$$

Multiplication Rule

- Given $P(E|F) = P(E \cap F) / P(F)$, multiplying both sides by $P(F)$, we obtain the multiplication rule

$$P(E \cap F) = P(E|F) \cdot P(F)$$

Example

- A card is drawn from a fair poker deck. What is the probability that the card is a face card given the card is a non-numbered card?

Example

- A card is drawn from a fair poker deck. What is the probability that the card is a face card given the card is a non-numbered card?
  - Since we know that the card is a non-numbered card, we need only consider the Ace, Jack, Queen, and King for each suit; there are 16 non-numbered cards.

Example

- A card is drawn from a fair poker deck. What is the probability that the card is a face card given the card is a non-numbered card?
  - Of the non-numbered cards, the Jack, Queen, and King are the face cards; there are 12 face cards among the 16 non-numbered cards.

Example

- A card is drawn from a fair poker deck. What is the probability that the card is a face card given the card is a non-numbered card?

P(face card| non-numbered card) = 12/16

= 3/4

Example

- A card is drawn from a fair poker deck. What is the probability that the card is red given the card is a seven?
- P(red|seven) = P(red seven)/P(seven)

  = [2/52]/[4/52]

  = 2/4

  = 1/2

Example

- A card is drawn from a fair poker deck. What is the probability that the card is red given the card is a seven?
- P(red|seven) = n(red seven)/n(seven)

  = 2/4

  = 1/2

Example

- A card is drawn from a fair poker deck. What is the probability that the card is a seven given the card is red?
- $P(seven|red) = P(red\ seven)/P(red)$

$$= (2/52)(26/52)$$

$$= 2/26$$

$$= 1/13$$

Example

- A card is drawn from a fair poker deck. What is the probability that the card is a seven given the card is red?
- P(seven|red) = n(red seven)/n(red)

  = 2/26

  = 1/13

Independent Events

- Two events E and F are independent
  - If the occurrence of event E does not affect the probability that event F occurs
  - If the occurrence that event F does not affect the probability that event E occurs

Independent Events

- Two events E and F are independent if and only if
  - $P(E|F) = P(E)$

or

  - $P(F|E) = P(F)$

Dependent Events

- Two events E and F are dependent if
  - The occurrence of event E affects the probability that event F occurs
  - The occurrence of event F affects the probability that event E occurs

Example

- Determine if the events E and F are dependent or independent. A card is drawn from a fair poker deck.
  - Event E: the card is a face card
  - Event F: the card is a lettered card

Example

- To determine if the events E and F are dependent or independent, we need to determine the following:
  - P(face card)
  - P(lettered card)
  - P(face card|lettered card)
  - P(lettered card|face card)

Example

- Event E: the card is a face card
- Event F: the card is a lettered card

  - P(face card) = 12/52 = 3/13
  - P(lettered card) = 16/52 = 4/13
  - P(face card|lettered card) = 12/16 = 3/4
  - P(lettered card|face card) = 12/12 = 1

Example

- P(face card) is not equal to
  P(face card|lettered card)
- P(lettered card) is not equal to
  P(lettered card|face card)

- So, the event *the card is a face card* and the event *the card is a lettered card* are not independent.
  - These events are dependent.

Example

- A card is drawn from a fair poker deck. What is the probability that the card is even numbered given the card is red?

Example

- A card is drawn from a fair poker deck. What is the probability that the card is even numbered given the card is red?

P(even numbered|red)

= n(even numbered red cards)/n(red cards)

= 10/26

= 5/13

Example

- Two cards are drawn without replacement from a fair poker deck. What is the probability that the first card is red and the second card is black?

Example

- Two cards are drawn without replacement from a fair poker deck. What is the probability that the first card is red and the second card is black?

P(first red and second black)

= P(first red) · P(second black|first red)

= (1/2) · (26/51)

= 13/51

Example

- Two cards are drawn without replacement from a fair poker deck. What is the probability that the first card is red and the second card is black?

P(first red and second black)

= P(first red) · P(second black|first red)

= (1/2) · (26/51)

= 13/51

<u>Note</u>: We used the Multiplication Rule

Drawing Cards One at a Time

- To determine the probability of an event that is determined by drawing three cards in a row without replacement, we take the product of the probabilities of the three events, keeping in mind that drawing the first card affects drawing the second card and drawing the first and second cards affect drawing the third card

$P(E \cap F \cap G)$

$$P(E \cap F \cap G) = P(E \cap F) \cdot P(G|E \cap F)$$
$$= P(E) \cdot P(F|E) \cdot P(G|E \cap F)$$

$P(E \cap F \cap G)$

$$P(E \cap F \cap G) = P(E \cap F) \cdot P(G|E \cap F)$$
$$= P(E) \cdot P(F|E) \cdot P(G|E \cap F)$$

Note: When we draw the first card, all the cards in the fair deck are available.

$P(E \cap F \cap G)$

$$P(E \cap F \cap G) = P(E \cap F) \cdot P(G|E \cap F)$$
$$= P(E) \cdot P(F|E) \cdot P(G|E \cap F)$$

<u>Note</u>: When we draw the second card, the first card has already been removed from the fair deck: there is one less card in the deck and we know what type of card has been removed.

$P(E \cap F \cap G)$

$$P(E \cap F \cap G) = P(E \cap F) \cdot P(G|E \cap F)$$
$$= P(E) \cdot P(F|E) \cdot P(G|E \cap F)$$

Note: When we draw the third card, the first and second cards have already been removed from the fair deck: there is two less cards in the deck and we know what type of cards have been removed.

Example

- Three cards are drawn without replacement from a fair poker deck. What is the probability that the first card is a King, the second card is a five, and the third card is a face card.

Example

- Three cards are drawn without replacement from a fair poker deck. What is the probability that the first card is a King, the second card is a five, and the third card is a face card.

P(K, five, face card)

$$= (4/52) \cdot (4/51) \cdot (11/50)$$
$$= (1/13) \cdot (2/51) \cdot (11/25)$$
$$= 22/16575$$

Example

P(K, five, face card)

$= (4/52) \cdot (4/51) \cdot (11/50)$

$= (1/13) \cdot (2/51) \cdot (11/25)$

$= 22/16575$

Notice that the denominator decreases for each fraction: the number of cards in the deck decreases by one each time a card is drawn.

Example

P(K, five, face card)

$= (4/52) \cdot (4/51) \cdot (11/50)$

$= (1/13) \cdot (2/51) \cdot (11/25)$

$= 22/16575$

Notice that the numerator for the last fraction is eleven rather than twelve: one of the twelve face cards has already been removed from the deck.